# DERIVATION OF AN APPROXIMATE THEORY OF BENDING OF A PLATE BY THE METHOD OF ASYMPTOTIC INTEGRATION OF THE EQUATIONS OF THE THEORY OF ELASTICITY

## (POSTROENIE PRIBLIZHENNOI TEORII IZGIBA PLASTINKI METODOM ASIMPTOTICHESKOGO INTEGRIROVANIIA URAVNENII TEORII UPRUGOSTI)

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The paper considers the possibility of making more exact the theory of plates based on Kirchhoff's hypothesis. The problem of the bending of a plate is formulated as a three-dimensional problem of the theory of elasticity which can be solved by an iteration process; it is assumed that one of the extensions of the region under consideration is small compared with the other two. The required state of stress of the plate is presented as the sum of a slowly damped state of stress, derived by means of a basic iteration process, and states of stress which are rapidly damped with increase in distance from the edge, and which are derived by means of auxiliary iteration processes. Such an approach is often used in the asymptotic integration of differential equations (see [1]) and corresponds to the physical nature of the problem. The basic iteration process enables us to find the state of stress which is given as a first approximation by the classical theory. The auxiliary iteration process allows us to take into account the stress distribution at the edges which were discussed in attempts to make the classical theory more exact by replacing Kirchhoff's hypothesis by alternative assumptions (see, for example, [2-6]).

**1.** It is required to solve the following system of differential equations of the theory of elasticity:

The equilibrium equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0 \qquad (xy) \qquad \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0 \qquad (1.1)$$

the formulas for the displacements and stresses

$$E\frac{\partial u}{\partial x} = \sigma_x - \nu(\sigma_y + \sigma_z) \quad (xy), \qquad E\frac{\partial w}{\partial z} = \sigma_z - \nu(\sigma_x + \sigma_y)$$
$$E\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = 2(1+\nu)\tau_{xz} \quad (xy), \qquad E\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 2(1+\nu)\tau_{xy} \tag{1.2}$$

Here and in what follows the symbol $(xy)$ will be used to denote the existence of a second relation derived from the given expression by replacing $x$ and $u$ by $y$ and $v$ and vice versa.

It is assumed that the $z$-axis of a Cartesian system of coordinates is perpendicular to the plane of the plate and that the $x$ and $y$ axes are situated in the middle plane of the plate. Since we are concerned with the bending of the plate, it is assumed throughout that the required state of stress and strain in the plate is skew-symmetrical about the $xoy$ plane.

We denote the thickness of the plate by $2h$ and then the following boundary conditions must be satisfied on the planes $z = \pm h$:

$$\sigma_z = \pm\frac{1}{2}p(x, y), \qquad \tau_{xz} = 0, \qquad \tau_{yz} = 0 \tag{1.3}$$

where $p(x, y)$ is the normal external load intensity.

We shall make use of the relation

$$\partial\sigma_z / \partial z = 0 \qquad \text{at } z = \pm h \tag{1.4}$$

which follows from (1.3) and the third of Equations (1.1).

The boundary conditions on the lateral surfaces of the plate will be formulated later.

**2.** A basic iteration process is defined as one which enables the basic states of stress (those that are not rapidly damped with increase in distance from an edge of the plate) to be found.

It is assumed that in the basic state of stress the stresses and displacements do not change too rapidly with respect to the variables $(x, y)$. In the $z$ direction these quantities obviously must vary rapidly. We shall therefore use the well-known method of scale extension and replace $z$ according to the formula

$$z = h\zeta \tag{2.1}$$

assuming that the rate of change of the stresses and displacements with respect to the variables $(x, y, \zeta)$ is not too high.

Equations (1.1) and (1.2) now become

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+h^{-1}\frac{\partial\tau_{xz}}{\partial\zeta}=0 \quad (xy), \qquad \frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+h^{-1}\frac{\partial\sigma_z}{\partial\zeta}=0$$
$$E\frac{\partial u}{\partial x}=\sigma_x-\nu(\sigma_y+\sigma_z) \quad (xy), \qquad Eh^{-1}\frac{\partial w}{\partial\zeta}=\sigma_z-\nu(\sigma_x+\sigma_y) \tag{2.2}$$
$$E\left(h^{-1}\frac{\partial u}{\partial\zeta}+\frac{\partial w}{\partial x}\right)=2(1+\nu)\tau_{xz} \quad (xy), \qquad E\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)=2(1+\nu)\tau_{xy}$$

If $Q$ is any one of the stresses or displacements, it can be expressed in the form

$$Q=h^{-q}\sum_{s=1}^{s=S}h^{s-1}Q^{(s)} \tag{2.3}$$

Here $q$ is an integer which is different for different displacements and stresses and which is defined by the following expressions:

$$(\sigma_x,\sigma_y,\tau_{xy})\to q=\varkappa+2, \qquad (\tau_{xz},\tau_{yz})\to q=\varkappa+1, \qquad \sigma_z\to q=\varkappa$$
$$(u,v)\to q=\varkappa+2, \qquad w\to q=\varkappa+3 \tag{2.4}$$

(the number $\kappa$ remaining for the present unspecified).

We express the stresses and displacements in Equations (2.2) in the form (2.3), (2.4) and equate coefficients of equal powers of $h$ on the left- and right-hand sides of each equation taken separately. We then obtain the system of equations

$$\frac{\partial\sigma_x^{(s)}}{\partial x}+\frac{\partial\tau_{xy}^{(s)}}{\partial y}+\frac{\partial\tau_{xz}^{(s)}}{\partial\zeta}=0 \quad (xy), \qquad \frac{\partial\tau_{xz}^{(s)}}{\partial x}+\frac{\partial\tau_{yz}^{(s)}}{\partial y}+\frac{\partial\sigma_z^{(s)}}{\partial\zeta}=0 \tag{2.5}$$
$$E\frac{\partial u^{(s)}}{\partial x}=\sigma_x^{(s)}-\nu\left(\sigma_y^{(s)}+\sigma_z^{(s-2)}\right) \quad (xy)$$
$$E\frac{\partial w^{(s)}}{\partial\zeta}=\sigma_z^{(s-4)}-\nu\left(\sigma_x^{(s-2)}+\sigma_y^{(s-2)}\right)$$
$$E\left(\frac{\partial u^{(s)}}{\partial\zeta}+\frac{\partial w^{(s)}}{\partial x}\right)=2(1+\nu)\tau_{xz}^{(s-2)} \quad (xy)$$
$$E\left(\frac{\partial u^{(s)}}{\partial y}+\frac{\partial v^{(s)}}{\partial x}\right)=2(1+\nu)\tau_{xy}^{(s)}$$

Here and in the future it is considered that the quantities $Q^{(s)}\equiv 0$ for $s<1$.

Equations (2.5) form a chain of systems of equations, and the basic iteration process consists of the successive determination (in order of magnitude of $s$) of $Q^{(s)}$ from the appropriate system. The value of $Q^{(s+1)}$ is determined from the values of $Q^{(1)}, Q^{(2)}, \ldots, Q^{(s)}$ already found.

3. The quantity $Q^{(s)}$, i.e. the solution of the system of equations (2.5), will be expressed as the sum of two terms $Q_i^{(s)} + Q^{*(s)}$. The first term represents the integral of the homogeneous system

$$\frac{\partial\sigma_x^{(s)}}{\partial x} + \frac{\partial\tau_{xy}^{(s)}}{\partial y} + \frac{\partial\tau_{xz}^{(s)}}{\partial\zeta} = 0 \quad (xy), \qquad \frac{\partial\tau_{xz}^{(s)}}{\partial x} + \frac{\partial\tau_{yz}^{(s)}}{\partial y} + \frac{\partial\sigma_z^{(s)}}{\partial\zeta} = 0$$

$$E\frac{\partial u^{(s)}}{\partial x} = \sigma_x^{(s)} - \nu\sigma_y^{(s)} \quad (xy), \qquad E\frac{\partial w^{(s)}}{\partial\zeta} = 0 \tag{3.1}$$

$$E\left(\frac{\partial u^{(s)}}{\partial\zeta} + \frac{\partial w^{(s)}}{\partial x}\right) = 0 \quad (xy), \qquad E\left(\frac{\partial u^{(s)}}{\partial y} + \frac{\partial v^{(s)}}{\partial x}\right) = 2(1+\nu)\tau_{xy}^{(s)}$$

obtained as a result of discarding from (2.5) quantities with a superscript less than $s$, and the second term represents a particular integral of the nonhomogeneous system (2.5) in which all the quantities with a superscript less than $s$ are considered to be known.

The system (3.1) can be easily integrated to give

$$u_i^{(s)} = \zeta u_1^{(s)} \quad (xy), \qquad w_i^{(s)} = w_0^{(s)}$$

$$\sigma_{xi}^{(s)} = \zeta\sigma_{x_1}^{(s)} \quad (xy), \qquad \tau_{xyi}^{(s)} = \zeta\tau_{xy1}^{(s)}, \qquad \tau_{xzi}^{(s)} = \zeta^2\tau_{xz2}^{(s)} + \tau_{xz0}^{(s)} \quad (xy)$$

$$\sigma_{zi} = \zeta^3\sigma_{z3}^{(s)} + \zeta\sigma_{z1}^{(s)} \tag{3.2}$$

Here the quantities distinguished by an additional numerical subscript (the subscripts are equal to the power of $\zeta$ by which this quantity is multiplied) are functions of two variables $(x, y)$ related by the following equalities:

$$u_1^{(s)} = -\frac{\partial w_0^{(s)}}{\partial x} \quad (xy), \qquad \sigma_{x1}^{(s)} = -\frac{E}{1-\nu^2}\left(\frac{\partial^2 w_0^{(s)}}{\partial x^2} + \nu\frac{\partial^2 w_0^{(s)}}{\partial ys^2}\right) \quad (xy)$$

$$\tau_{xy1}^{(s)} = -\frac{E}{1+\nu}\frac{\partial^2 w_0^{(s)}}{\partial x\partial y}, \qquad \tau_{xz2}^{(s)} = -\frac{1}{2}\left(\frac{\partial\sigma_{x1}^{(s)}}{\partial x} + \frac{\partial\tau_{xy1}^{(s)}}{\partial y}\right) \quad (xy) \tag{3.3}$$

$$\sigma_{z3}^{(s)} = \frac{1}{3!}\left(\frac{\partial^2\sigma_{x1}^{(s)}}{\partial x^2} + 2\frac{\partial^2\tau_{xy1}^{(s)}}{\partial x\partial y} + \frac{\partial^2\sigma_{y1}^{(s)}}{\partial y^2}\right), \qquad \sigma_{z1}^{(s)} = -\left(\frac{\partial\tau_{xz0}^{(s)}}{\partial x} + \frac{\partial\tau_{yz0}^{(s)}}{\partial y}\right)$$

The integral of equations (2.5) can be written as follows:

$$Ew^{*(s)} = \int_0^\zeta [\sigma_z^{(s-4)} - \nu(\sigma_x^{(s-2)} + \sigma_y^{(s-2)})]\,d\zeta$$

$$Eu^{*(s)} = 2(1+\nu)\int_0^\zeta \tau_{xz}^{(s-2)}\,d\zeta - E\frac{\partial}{\partial x}\int_0^\zeta w^{*(s)}d\zeta\,. \quad (xy) \tag{3.4}$$

$$\sigma_x^{*(s)} = \frac{E}{1-\nu^2}\left(\frac{\partial u^{*(s)}}{\partial x} + \nu\frac{\partial v^{*(s)}}{\partial y}\right) + \frac{\nu}{1-\nu}\sigma_z^{(s-2)} \quad (xy)$$

$$\tau_{xy}^{*(s)} = \frac{E}{2(1+\nu)}\left(\frac{\partial u^{*(s)}}{\partial y} + \frac{\partial v^{*(s)}}{\partial x}\right)$$

$$\tau_{xz}^{*(s)} = -\int_0^\zeta \left(\frac{\partial \sigma_x^{*(s)}}{\partial x} + \frac{\partial \tau_{xy}^{*(s)}}{\partial y}\right) d\zeta \quad (xy), \qquad \sigma_z^{*(s)} = -\int_0^\zeta \left(\frac{\partial \tau_{xz}^{*(s)}}{\partial x} + \frac{\partial \tau_{yz}^{*(s)}}{\partial y}\right) d\zeta$$

Here the quantities distinguished by an asterisk are functions of the variables $(x, y, \zeta)$; those without the asterisk and with a superscript less than $s$ are considered to be known quantities.

As has already been pointed out, $Q^{(s)} \equiv 0$ for $s < 1$. Therefore, it follows from (3.4) that $Q^{*(1)}$ and $Q^{*(2)}$, quantities with an asterisk and with $s = 1$ and $s = 2$, are identically zero. For $s > 2$ these quantities are polynomials in the variable $\zeta$ and can be found from the recurrence formulas (3.4).

It is required that the stresses found by means of the basic iteration process satisfy the boundary conditions (1.3) and (1.4). More precisely, we shall impose the even more stringent requirements that

$$\left.\begin{array}{l} \sigma_z^{(1)} = \frac{1}{2} p, \quad \sigma_z^{(s)} = 0 \quad (s > 1) \\ \partial \sigma_z^{(s)} / \partial z = 0, \quad \tau_{xz}^{(s)} = 0, \quad \tau_{yz}^{(s)} = 0, \quad (s \geqslant 1) \end{array}\right\} \text{at} \quad z = h\,(\zeta = 1)$$

Analogous conditions will be satisfied automatically at $z = -h$, since the problem is skew-symmetrical.

With the aid of (2.3), (2.4) and (3.2) we obtain

$$h^{s-\varkappa-1}(\sigma_{z3}^{(s)} + \sigma_{z1}^{(s)} + \sigma_z^{*(s)}) = \frac{1}{2} p_s, \qquad h^{s-\varkappa-2}\left(3\sigma_{z3}^{(s)} + \sigma_{z1}^{(s)} + \frac{\partial \sigma_z^{*(s)}}{\partial \zeta}\right) = 0$$

$$h^{s-\varkappa-2}(\tau_{xz2}^{(s)} + \tau_{xz0}^{(s)} + \tau_{xz}^{*(s)}) = 0 \quad (xy) \tag{3.5}$$

Here $p_1 = p$ and $p_s = 0$ for $s > 1$.

It is assumed that $p(x, y)$ is independent of $h$ and that the quantities $Q^{(s)}$ and $Q^{*(s)}$ also need not depend on $h$. If we satisfy this requirement and put $\varkappa = 0$ we can solve equations (3.5) for $\sigma_{z3}^{(s)}$, $\sigma_{z1}^{(s)}$, $\tau_{xz0}^{(s)}$, $\tau_{yz0}^{(s)}$. Bearing in mind that quantities distinguished by an asterisk are nonzero only for $s > 2$, we find that

$$\sigma_{z3}^{(1)} = -\frac{1}{4} p, \qquad \sigma_{z1}^{(1)} = \frac{3}{4} p$$

$$\sigma_{z3}^{(s)} = \frac{1}{2}\left(\sigma_z^{*(s)} - \frac{\partial \sigma_z^{*(s)}}{\partial \zeta}\right), \qquad \sigma_{z1}^{(s)} = -\frac{1}{2}\left(3\sigma_z^{*(3)} - \frac{\partial \sigma_z^{*(s)}}{\partial \zeta}\right)(s>1) \quad (3.6)$$

$$\tau_{xz0}^{(s)} = -\tau_{xz2}^{(s)} - \tau_{xz}^{*(s)} \qquad (xy)$$

Thus $\sigma_{z3}^{(1)}$ can be expressed in terms of $p$, $\sigma_{z3}^{(2)} = 0$, and for $s > 2$, $\sigma_{z3}$ can be expressed in terms of quantities distinguished by the superscript $(s)$ and an asterisk. This means that for any $s$ we can consider $\sigma_{z3}^{(s)}$ to be a known quantity if $s - 2$ first approximations have been found, and consequently, (3.3) forms a system from which

$$u_1^{(s)}, \quad v_1^{(s)}, \quad w_0^{(s)}, \quad \sigma_{x1}^{(s)}, \quad \sigma_{y1}^{(s)}, \quad \tau_{xy1}^{(s)}, \quad \tau_{xz2}^{(s)}, \quad \tau_{yz2}^{(s)}$$

can be determined.

It will be noticed that this system reduces to a nonhomogeneous (for $s > 2$) biharmonic equation (with respect to $x$, $y$) in $w_0^{(s)}$.

**4.** Let us turn now to the auxiliary iteration process, i.e. the process by means of which we find the states of stress which are damped, no matter how rapidly (for a sufficiently small $h$), with increase in distance from some fixed line in the plane $xoy$. In order to simplify the computations, it is assumed throughout that this line is $x = 0$ and that damping takes place on the side $x < 0$.

If in (1.1) and (1.2) we make the substitutions

$$x = h\xi, \qquad z = h\zeta$$

the seconf of which is the same as (2.1), we obtain

$$\begin{gathered}
h^{-1}\frac{\partial \sigma_x}{\partial \xi} + \frac{\partial \tau_{xy}}{\partial y} + h^{-1}\frac{\partial \tau_{xz}}{\partial \zeta} = 0, \qquad Eh^{-1}\frac{\partial u}{\partial \xi} = \sigma_x - \nu(\sigma_y + \sigma_z) \\
h^{-1}\frac{\partial \tau_{xy}}{\partial \xi} + \frac{\partial \sigma_y}{\partial y} + h^{-1}\frac{\partial \tau_{yz}}{\partial \zeta} = 0, \qquad E\frac{\partial v}{\partial y} = \sigma_y - \nu(\sigma_z + \sigma_x) \\
h^{-1}\frac{\partial \tau_{xz}}{\partial \xi} + \frac{\partial \tau_{yz}}{\partial y} + h^{-1}\frac{\partial \sigma_z}{\partial \zeta} = 0, \qquad Eh^{-1}\frac{\partial w}{\partial \zeta} = \sigma_z - \nu(\sigma_x + \sigma_y) \qquad (4.1) \\
E\left(h^{-1}\frac{\partial v}{\partial \zeta} + \frac{\partial u}{\partial y}\right) = 2(1+\nu)\tau_{yz}, \qquad Eh^{-1}\left(\frac{\partial u}{\partial \zeta} + \frac{\partial w}{\partial \xi}\right) = 2(1+\nu)\tau_{xz} \\
E\left(\frac{\partial u}{\partial y} + h^{-1}\frac{\partial v}{\partial \xi}\right) = 2(1+\nu)\tau_{xy}
\end{gathered}$$

It will be assumed that the rate of variation of the unknown quantities with respect to the variables $(\xi, y, \zeta)$ is not too high.

We denote any of the unknown stresses or displacements by $R$, where $R$ is defined as

$$R = h^r \sum_{s=1}^{s=S} h^{s-1} R^{(s)} \tag{4.2}$$

Here $r$ has different values for different stresses and displacements, and there are two possible variants for the choice of values of $r$.

The first variant is given by

$$\begin{gathered}(\tau_{xy}, \tau_{yz}) \to r = -\lambda, \qquad (\sigma_x, \sigma_y, \tau_{xz}, \sigma_z) \to r = -\lambda + 1 \\ (u, w) \to r = -\lambda + 2, \qquad v \to r = -\lambda + 1\end{gathered} \tag{4.3}$$

and the second by

$$\begin{gathered}(\sigma_x, \sigma_y, \tau_{xz}, \sigma_z) \to r = -\mu + 1, \qquad (\tau_{xy}, \tau_{yz}) \to r = -\mu + 2 \\ (u, w) \to r = -\mu + 2, \qquad v \to r = -\mu + 3\end{gathered} \tag{4.4}$$

where $\lambda$ and $\mu$ are for the present arbitrary numbers.

Correspondingly, it is possible to formulate two variants of the auxiliary iteration process. In order to do so we substitute expansions (4.2) into equations (4.1), express $r$ according to formulas (4.3) or (4.4) and then equate coefficients of equal powers of $h$ on the right- and left-hand sides of each equality taken separately. In this way we obtain

$$\begin{gathered}\frac{\partial \sigma_x^{(\alpha)}}{\partial \xi} + \frac{\partial \tau_{xy}^{(\beta)}}{\partial y} + \frac{\partial \tau_{xz}^{(\alpha)}}{\partial \zeta} = 0, \qquad E\,\frac{\partial u^{(\alpha)}}{\partial \xi} = \sigma_x^{(\alpha)} - \nu\,(\sigma_y^{(\alpha)} + \sigma_z^{(\alpha)}) \\ \frac{\partial \tau_{xy}^{(\alpha)}}{\partial \xi} + \frac{\partial \sigma_y^{(\gamma)}}{\partial y} + \frac{\partial \tau_{yz}^{(\alpha)}}{\partial \zeta} = 0, \qquad E\,\frac{\partial v^{(\beta)}}{\partial y} = \sigma_y^{(\alpha)} - \nu\,(\sigma_z^{(\alpha)} + \sigma_x^{(\alpha)}) \\ \frac{\partial \tau_{xz}^{(\alpha)}}{\partial \xi} + \frac{\partial \tau_{yz}^{(\beta)}}{\partial y} + \frac{\partial \sigma_z^{(\alpha)}}{\partial \zeta} = 0, \qquad E\,\frac{\partial w^{(\alpha)}}{\partial \zeta} = \sigma_z^{(\alpha)} - \nu\,(\sigma_x^{(\alpha)} + \sigma_y^{(\alpha)}) \\ E\left(\frac{\partial v^{(\alpha)}}{\partial \zeta} + \frac{\partial w^{(\gamma)}}{\partial y}\right) = 2\,(1+\nu)\,\tau_{yz}^{(\alpha)}, \qquad E\left(\frac{\partial u^{(\alpha)}}{\partial \zeta} + \frac{\partial w^{(\alpha)}}{\partial \xi}\right) = 2\,(1+\nu)\,\tau_{xz}^{(\alpha)} \\ E\left(\frac{\partial u^{(\gamma)}}{\partial y} + \frac{\partial v^{(\alpha)}}{\partial \xi}\right) = 2\,(1+\nu)\,\tau_{xy}^{(\alpha)}\end{gathered} \tag{4.5}$$

Here $\alpha$, $\beta$, $\gamma$ are defined by

$$\alpha = s, \qquad \beta = s, \qquad \gamma = s - 2 \quad \text{for the first variant} \tag{4.6}$$

$$\alpha = s, \qquad \beta = s - 2, \qquad \gamma = s \quad \text{for the second variant} \tag{4.7}$$

In equations (4.5), as before, $R^{(s)} \equiv 0$ for $s < 1$.

5. Let us make a more detailed examination of the first variant of the auxiliary iteration process. According to (4.6) we can express the system of equations (4.5) in the form

$$\frac{\partial \tau_{xy}^{(s)}}{\partial \xi} + \frac{\partial \sigma_y^{(s-2)}}{\partial y} + \frac{\partial \tau_{yz}^{(s)}}{\partial \zeta} = 0 \tag{5.1}$$

$$E\left(\frac{\partial v^{(s)}}{\partial \zeta} + \frac{\partial w^{(s-2)}}{\partial y}\right) = 2(1+\nu)\,\tau_{yz}^{(s)}, \qquad E\left(\frac{\partial v^{(s)}}{\partial \xi} + \frac{\partial u^{(s-2)}}{\partial y}\right) = 2(1+\nu)\tau_{xy}^{(s)}$$

$$\frac{\partial \sigma_x^{(s)}}{\partial \xi} + \frac{\partial \tau_{xy}^{(s)}}{\partial y} + \frac{\partial \tau_{xz}^{(s)}}{\partial \zeta} = 0, \qquad \frac{\partial \tau_{xz}^{(s)}}{\partial \xi} + \frac{\partial \tau_{yz}^{(s)}}{\partial y} + \frac{\partial \sigma_z^{(s)}}{\partial \zeta} = 0 \tag{5.2}$$

$$E\,\frac{\partial u^{(s)}}{\partial \xi} = \sigma_x^{(s)} - \nu\left(\sigma_y^{(s)} + \sigma_z^{(s)}\right), \qquad E\,\frac{\partial v^{(s)}}{\partial y} = \sigma_y^{(s)} - \nu\left(\sigma_z^{(s)} + \sigma_x^{(s)}\right)$$

$$E\,\frac{\partial w^{(s)}}{\partial \zeta} = \sigma_z^{(s)} - \nu\left(\sigma_x^{(s)} + \sigma_y^{(s)}\right), \qquad E\left(\frac{\partial u^{(s)}}{\partial \zeta} + \frac{\partial w^{(s)}}{\partial \xi}\right) = 2(1+\nu)\,\tau_{xz}^{(s)}$$

The solution of equations (5.1) and (5.2) can be expressed by

$$R^{(s)} = R_{\mathrm{I}}^{(s)} + R_{\mathrm{I}}^{*(s)} \tag{5.3}$$

and we shall assume that:

$(\tau_{xy\mathrm{I}}^{(s)}, \tau_{yz\mathrm{I}}^{(s)}, v_{\mathrm{I}}^{(s)})$ is the general solution of the homogeneous system obtained by equating to zero quantities with a superscript $(s-2)$ in (5.1);

$(\sigma_{x\mathrm{I}}^{(s)}, \tau_{xz\mathrm{I}}^{(s)}, \sigma_{y\mathrm{I}}^{(s)}, \sigma_{z\mathrm{I}}^{(s)}, u_{\mathrm{I}}^{(s)}, w_{\mathrm{I}}^{(s)})$ is the particular integral of the nonhomogeneous system of equations obtained by putting

$$\tau_{xy}^{(s)} = \tau_{xy\mathrm{I}}^{(s)}, \qquad \tau_{yz}^{(s)} = \tau_{yz\mathrm{I}}^{(s)}, \qquad v^{(s)} = v_{\mathrm{I}}^{(s)}$$

in (5.2) and assuming that these quantities are known;

$(\tau_{xy\mathrm{I}}^{*(s)}, \tau_{yz\mathrm{I}}^{*(s)}, v_{\mathrm{I}}^{*(s)})$ is the particular integral of the nonhomogeneous system (5.1), in which the quantities with a superscript $(s-2)$ are considered to be known;

$(\sigma_{x\mathrm{I}}^{*(s)}, \tau_{xz\mathrm{I}}^{*(s)}, \sigma_{y\mathrm{I}}^{*(s)}, \sigma_{z\mathrm{I}}^{*(s)}, u_{\mathrm{I}}^{*(s)}, w_{\mathrm{I}}^{*(s)})$ is the particular integral of the nonhomogeneous system (5.2) obtained by putting

$$\tau_{xy}^{(s)} = \tau_{xyI}^{*(s)}, \qquad \tau_{yz}^{(s)} = \tau_{yz\mathrm{I}}^{*(s)}, \qquad v^{(s)} = v_{\mathrm{I}}^{*(s)}$$

and assuming that they are known quantities.

Since $R^{(s-2)} = 0$ for $s = 1, 2$, we can put $R_{\mathrm{I}}^{*(1)} = 0$, $R_{\mathrm{I}}^{*(2)} = 0$.

In the first variant of the auxiliary iteration process the basic system of equations is the one which determines $(\tau_{xyI}^{(s)}, \tau_{yIz}^{(s)}, v_I^{(s)})$. It is the system of differential equations obtained in the problem of torsion of a prismatic rod (with its axis along the $y$ axis).

6. The solution denoted by $R_I^{(s)}$ in Formula (5.3) is obtained by integration of a harmonic equation. Indeed, if we discard quantities with a superscript $(s - 2)$ in (5.1), all these equations can be satisfied by putting

$$\tau_{xy}^{(s)} = \tau_{xyI}^{(s)} = \frac{\partial^2\Psi^{(s)}}{\partial\xi^2}, \quad \tau_{yz}^{(s)} = \tau_{yzI}^{(s)} = \frac{\partial^2\Psi^{(s)}}{\partial\xi\partial\zeta}, \quad Ev^{(s)} = Ev_I^{(s)} = 2(1+\nu)\frac{\partial\Psi^{(s)}}{\partial\xi} \tag{6.1}$$

where $\Psi^{(s)}$ is a harmonic function of the variables $\xi$, $\zeta$

$$\frac{\partial^2\Psi^{(s)}}{\partial\xi^2} + \frac{\partial^2\Psi^{(s)}}{\partial\zeta^2} = 0 \tag{6.2}$$

Equations (5.2) will also be satisfied if we put

$$\begin{gathered}\sigma_x^{(s)} = \sigma_{xI}^{(s)} = -2\frac{\partial^2\Psi^{(s)}}{\partial\xi\partial y}, \qquad \tau_{xz}^{(s)} = \tau_{xzI}^{(s)} = -\frac{\partial^2\Psi^{(s)}}{\partial y\,\partial\zeta} \\ \sigma_z^{(s)} = \sigma_{zI}^{(s)} = 0, \qquad \sigma_y^{(s)} = \sigma_{yI}^{(s)} = 2\frac{\partial^2\Psi^{(s)}}{\partial\xi\,\partial y} \\ Eu^{(s)} = Eu_I^{(s)} = -2(1+\nu)\frac{\partial\Psi^{(s)}}{\partial y}, \qquad Ew^{(s)} = Ew_I^{(s)} = 0\end{gathered} \tag{6.3}$$

7. Let us now consider the second variant of the auxiliary iteration process; according to (4.7), Equations (4.5) can be written as follows:

$$\begin{gathered}\frac{\partial\sigma_x^{(s)}}{\partial\xi} + \frac{\partial\tau_{xy}^{(s-2)}}{\partial y} + \frac{\partial\tau_{xz}^{(s)}}{\partial\zeta} = 0, \qquad \frac{\partial\tau_{xz}^{(s)}}{\partial\xi} + \frac{\partial\tau_{yz}^{(s-2)}}{\partial y} + \frac{\partial\sigma_z^{(s)}}{\partial\zeta} = 0 \\ E\frac{\partial u^{(s)}}{\partial\xi} = \sigma_x^{(s)} - \nu(\sigma_y^{(s)} + \sigma_z^{(s)}), \qquad E\frac{\partial v^{(s-2)}}{\partial y} = \sigma_y^{(s)} - \nu(\sigma_z^{(s)} + \sigma_x^{(s)}) \\ E\frac{\partial w^{(s)}}{\partial\zeta} = \sigma_z^{(s)} - \nu(\sigma_x^{(s)} + \sigma_y^{(s)}), \qquad E\left(\frac{\partial u^{(s)}}{\partial\zeta} + \frac{\partial w^{(s)}}{\partial\xi}\right) = 2(1+\nu)\tau_{xz}^{(s)}\end{gathered} \tag{7.1}$$

$$\begin{gathered}\frac{\partial\tau_{xy}^{(s)}}{\partial\xi} + \frac{\partial\sigma_y^{(s)}}{\partial y} + \frac{\partial\tau_{yz}^{(s)}}{\partial\zeta} = 0 \\ E\left(\frac{\partial v^{(s)}}{\partial\zeta} + \frac{\partial w^{(s)}}{\partial y}\right) = 2(1+\nu)\tau_{yz}^{(s)}, \qquad E\left(\frac{\partial v^{(s)}}{\partial\xi} + \frac{\partial u^{(s)}}{\partial y}\right) = 2(1+\nu)\tau_{xy}^{(s)}\end{gathered} \tag{7.2}$$

We can express the solution to this system of equations in the form

$$R^{(s)} = R_{II}^{(s)} + R_{II}^{*(s)} \tag{7.3}$$

and we shall assume that:

$(\sigma_{xII}^{(s)}, \tau_{xzII}^{(s)}, \sigma_{yII}^{(s)}, \sigma_{zII}^{(s)}, u_{II}^{(s)}, w_{II}^{(s)})$ is the general solution of the homogeneous system of equations obtained by equating to zero quantities with a superscript $(s-2)$ in (7.1);

$(\tau_{xyII}^{(s)}, \tau_{yzII}^{(s)}, v_{II}^{(s)})$ is the particular integral of the nonhomogeneous system of equations obtained by putting

$$\sigma_x^{(s)} = \sigma_{xII}^{(s)}, \quad \tau_{xz}^{(s)} = \tau_{xzII}^{(s)}, \quad \sigma_z^{(s)} = \sigma_{zII}^{(s)}, \quad \sigma_y^{(s)} = \sigma_{yII}^{(s)}$$
$$u^{(s)} = u_{II}^{(s)}, \quad w^{(s)} = w_{II}^{(s)}$$

in (7.2) and assuming these to be known quantities;

$(\sigma_{xII}^{*(s)}, \tau_{xzII}^{*(s)}, \sigma_{yII}^{*(s)}, \sigma_{zII}^{*(s)}, u_{II}^{*(s)}, w_{II}^{*(s)})$ is the particular integral of the nonhomogeneous system (7.1) in which quantities with a superscript $(s-2)$ are considered to be known;

$(\tau_{xyII}^{*(s)}, \tau_{yzII}^{*(s)}, v_{II}^{*(s)})$ is the particular integral of the nonhomogeneous system of equations obtained by putting

$$\sigma_x^{(s)} = \sigma_{xII}^{*(s)}, \quad \tau_{xz}^{(s)} = \tau_{xzII}^{*(s)}, \quad \sigma_y^{(s)} = \sigma_{yII}^{*(s)}, \quad \sigma_z^{(s)} = \sigma_{zII}^{*(s)}$$
$$u^{(s)} = u_{II}^{*(s)}, \quad w^{(s)} = w_{II}^{*(s)}$$

in (7.2) and considering these as known quantities.

The quantities $R^{(1)}$ and $R^{(2)}$ are zero and we can therefore accept that $R_{II}^{*(1)}$ and $R_{II}^{*(2)}$ are also zero.

In the second variant of the auxiliary iteration process the basic system of equations is the one which determines

$$\sigma_{xII}^{(s)}, \tau_{xzII}^{(s)}, \sigma_{yII}^{(s)}, \sigma_{zII}^{(s)}, u_{II}^{(s)}, w_{II}^{(s)}$$

It is the system of differential equations obtained in the problem of plane deformation (in the plane $xOz$).

**8.** The solution denoted by $R_{II}^{(s)}$ in formula (7.3) is obtained by integrating a biharmonic equation. Indeed, if we discard all the quantities with a superscript $(s-2)$ in (7.1), we can satisfy all these equations by putting

$$\sigma_x^{(s)} = \sigma_{xII}^{(s)} = \frac{\partial^3\Phi^{(s)}}{\partial\xi\partial\zeta^2}, \qquad \sigma_z^{(s)} = \sigma_{zII}^{(s)} = \frac{\partial^3\Phi^{(s)}}{\partial\xi^3}$$

$$\sigma_y^{(s)} = \sigma_{yII}^{(s)} = \nu\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\zeta^2}\right)\frac{\partial\Phi^{(s)}}{\partial\xi}, \qquad \tau_{xz}^{(s)} = \tau_{xzII}^{(s)} = -\frac{\partial^3\Phi^{(s)}}{\partial\xi^2\partial\zeta} \tag{8.1}$$

$$Eu^{(s)} = Eu_{II}^{(s)} = (1-\nu^2)\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\zeta^2}\right)\Phi^{(s)} - (1+\nu)\frac{\partial^2\Phi^{(s)}}{\partial\xi^2}$$

$$Ew^{(s)} = Ew_{\text{II}}^{(s)} = (1 - \nu^2)\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\zeta^2}\right)\int\frac{\partial\Phi^{(s)}}{\partial\xi}\,d\zeta - (1+\nu)\frac{\partial^2\Phi^{(s)}}{\partial\xi\partial\zeta}$$

where $\Phi^{(s)}$ is a biharmonic function of the variables $\xi$, $\zeta$

$$\frac{\partial^4\Phi^{(s)}}{\partial\xi^4} + 2\frac{\partial^4\Phi^{(s)}}{\partial\xi^2\partial\zeta^2} + \frac{\partial^4\Phi^{(s)}}{\partial\zeta^4} = 0 \tag{8.2}$$

Equations (7.2) can also be satisfied if we put

$$\tau_{xy}^{(s)} = \tau_{xy\text{II}}^{(s)} = \frac{\partial^3\Phi^{(s)}}{\partial y\partial\zeta^2} - \nu\frac{\partial}{\partial y}\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\zeta^2}\right)\Phi^{(s)}, \quad \tau_{yz}^{(s)} = \tau_{yz\text{II}}^{(s)} = -\frac{\partial^3\Phi^{(s)}}{\partial\xi\partial y\partial\zeta}$$
$$Ev^{(s)} = Ev_{\text{II}}^{(s)} = (1-\nu^2)\frac{\partial}{\partial y}\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\zeta^2}\right)\int\Phi^{(s)}d\xi - (1+\nu)\frac{\partial^2\Phi^{(s)}}{\partial\xi\partial y} \tag{8.3}$$

**9.** We shall make the stipulation that in both variants of the auxiliary iteration process the homogeneous boundary conditions (1.3) must be satisfied in each approximation. In this way conditions (1.3) will be satisfied for the sum of integrals corresponding to all three iteration processes.

It will be assumed that in formulas (5.3) and (7.3) the particular integrals $R_{\text{I}}^{*(s)}$ and $R_{\text{II}}^{*(s)}$ are chosen such that they both satisfy the homogeneous boundary conditions (1.3) for any value of $s$ and both are damped with increase in distance from $\xi = 0$ in the direction $\xi < 0$.

The determination of $R_{\text{I}}^{*(s)}$ and $R_{\text{II}}^{*(s)}$, each taken separately, reduces to the integration of a system of equations equivalent to a single Poisson equation and to a nonhomogeneous biharmonic equation. It is therefore assumed that there are sufficient arbitrary constants of integration to satisfy all these requirements, since no conditions are imposed on $R_{\text{I}}^{*(s)}$ and $R_{\text{II}}^{*(s)}$ on the edge $\xi = 0$.

With such a choice of $R_{\text{I}}^{*(s)}$ and $R_{\text{II}}^{*(s)}$ the integrals $R_{\text{I}}^{(s)}$ and $R_{\text{II}}^{(s)}$ in (5.3) and (7.3) must also satisfy the homogeneous boundary conditions (1.3) and the damping condition as $\xi \to -\infty$. Furthermore, in deriving $R_{\text{I}}^{(s)}$ and $R_{\text{II}}^{(s)}$ certain arbitrary constants must be retained for satisfying the boundary conditions on the edges of the plate.

In addition to the homogeneous boundary conditions and the damping conditions, we shall therefore make the further requirement that:

for $R_{\text{I}}^{(s)}$ one boundary condition of the form

$$\Gamma_{\text{I}}^{(s)} = \gamma_{\text{I}}^{(s)} \qquad \text{at} \ \ \xi = 0 \tag{9.1}$$

and for $R_{II}{}^{(s)}$ two boundary conditions of the form

$$\Gamma_{IIk}^{(s)} = \gamma_{IIk}^{(s)} \qquad \text{for } \xi = 0 \qquad (k = 1,2) \tag{9.2}$$

are satisfied.

Here $\Gamma_I{}^{(s)}$ and $\Gamma_{IIk}{}^{(s)}$ are homogeneous linear functions of quantities which have been denoted by $R_I{}^{(s)}$ and $R_{II}{}^{(s)}$, respectively, and $\gamma_I{}^{(s)}$ and $\gamma_{IIk}{}^{(s)}$ are arbitrary functions of the variable $(y, \zeta)$.

The quantities $R_I{}^{(s)}$ are defined by formulas (6.1) and (6.3). It is easy to see that they will satisfy the homogeneous boundary conditions (1.3) provided that $\Psi^{(s)}$ satisfies the boundary conditions

$$\frac{\partial \Psi^{(s)}}{\partial \zeta} = 0 \qquad \text{for } \zeta = \pm 1 \tag{9.3}$$

in addition to the equation (6.2).

From now on we shall be concerned with only two variants of conditions (9.1), namely

$$\tau_{xyI}^{(s)} = \gamma_I, \qquad v_I{}^{(s)} = \gamma_I \tag{9.4}$$

It can easily be shown that Equation (6.2) has a (unique) solution which at $\zeta = \pm 1$ satisfies condition (9.3), is damped as $\xi \to -\infty$, and which at $\xi = 0$ satisfies one of the conditions (9.4) whatever the function $\gamma_I$ (which, of course, possesses certain properties of continuity).

It is possible to obtain this solution by the method of separation of variables.

For $R_{II}{}^{(s)}$, however, the question of the existence of such solutions is far more complex. The quantities $R_{II}{}^{(s)}$ are defined by Formulas (8.1) and (8.3). It will readily be seen that they will satisfy the homogeneous boundary conditions (1.3) provided $\Phi^{(s)}$ satisfies the boundary conditions

$$\Phi^{(s)} = \frac{\partial \Phi^{(s)}}{\partial \zeta} = 0 \qquad \text{for } \zeta = \pm 1 \tag{9.5}$$

in addition to Equation (8.2).

In order that Equation (8.2) has a solution which at $\zeta = \pm 1$ satisfies the conditions (9.5), is subject to damping as $\xi \to -\infty$ and which at $\xi = 0$ satisfies the two conditions (9.2), it is necessary to satisfy certain compatibility relations, the significance of which will be explained in Section 10, between conditions (9.5) and (9.2).

**10.** If we discard terms with a superscript $(s - 2)$ in the first two equalities of (7.1), we obtain the two equations

$$\frac{\partial \sigma_x{}^{(s)}}{\partial \xi} + \frac{\partial \tau_{xz}^{(s)}}{\partial \zeta} = 0, \qquad \frac{\partial \tau_{xz}^{(s)}}{\partial \xi} + \frac{\partial \sigma_z{}^{(s)}}{\partial \zeta} = 0 \tag{10.1}$$

which, amongst others, must be satisfied by the quantities $R_{II}{}^{(s)}$.

We carry out the following integration operations on equalities (10.1), the first operation being performed on the first equality and the second operation being performed on the second equality:

$$\int_{-\infty}^{0} d\xi \int_{-1}^{+1} d\zeta \int_{-1}^{\zeta} d\zeta, \qquad \int_{-\infty}^{0} d\xi \int_{-\infty}^{\xi} d\xi \int_{-1}^{+1} d\zeta$$

Where necessary the order of integration will be altered and the differential sign will be taken out from under the integral sign. Then, taking into account the homogeneous boundary conditions (1.3) and considering that the damping condition can be treated as the requirement that at $\xi = -\infty$ all the stresses and displacements vanish, we obtain: from the first of equalities (10.1)

$$\int_{-1}^{+1} d\zeta \int_{-1}^{\zeta} \sigma_{xII}^{(s)} \Big|_{\xi=0} d\zeta = 0, \quad \text{or} \quad \int_{-1}^{+1} \zeta \sigma_{xII}^{(s)} \Big|_{\xi=0} d\zeta = 0 \tag{10.2}$$

If we carry out the operation $\int_{-\infty}^{0} d\xi \int_{-1}^{+1} d\zeta$ on the second of equalities (10.1), we find that

$$\int_{-1}^{+1} \tau_{xzII}^{(s)} \Big|_{\xi=0} d\zeta = 0 \tag{10.3}$$

By virtue of (1.3) we have that

$$\tau_{xz}^{(s)} \Big|_{\xi=0} = \int_{-1}^{\zeta} \frac{\partial \tau_{xz}^{(s)}}{\partial \zeta} \Big|_{\xi=0} d\zeta$$

and consequently,

$$\int_{-1}^{+1} d\zeta \int_{-1}^{\zeta} \frac{\partial \tau_{xz}^{(s)}}{\partial \zeta} \Big|_{\xi=0} d\zeta = 0 \tag{10.4}$$

In certain cases equalities (10.2) to (10.4) constitute the compatibility relations. They indicate the requirements that must be imposed on the boundary values (at $\xi = 0$) of $\sigma_{xII}{}^{(s)}$ and $\tau_{xzII}{}^{(s)}$, if these quantities are given by boundary conditions (9.2), in order that a solution to the problem of finding $R_{II}{}^{(s)}$ can be obtained in which no discontinuities occur at the edges of the plate ($\xi = 0$, $\zeta = \pm 1$).

*Note.* Equalities (10.2) and (10.3) have a simple physical meaning. The first expresses the vanishing of the bending moment produced by the normal stresses $\sigma_{xII}^{(s)}$ on the edges, and the second expresses the vanishing of the shear force created by the shear stresses $\tau_{xzII}^{(s)}$ on the edges.

We shall denote the compatibility relations by

$$A\,[\varphi\,(y,\,\zeta);\,P] = 0 \tag{10.5}$$

taking this as a relation which the function $\varphi(y, \zeta)$ must satisfy if it has the boundary value $P$. For example, with the aid of (10.2), we can rewrite

$$A\,[\varphi;\,\sigma_{xII}^{(s)}] = 0$$

as follows:

$$\int_{-1}^{+1} \zeta\varphi\,d\zeta = 0$$

The symbolic notation of (10.5) will be used in those cases when $P = u_{II}^{(s)}$ or $P = w_{II}^{(s)}$, although the question of how these equalities can be made more specific is still unsolved.

**11.** Let us consider the following five variants of the boundary conditions on the edges of the plate:

$$\sigma_x = 0, \quad \tau_{xy} = 0, \quad \tau_{xz} = 0 \tag{11.1}$$

$$u = 0, \quad v = 0, \quad w = 0 \tag{11.2}$$

$$u = 0, \quad \tau_{xy} = 0, \quad w = 0 \tag{11.3}$$

$$\sigma_x = 0, \quad v = 0, \quad w = 0 \tag{11.4}$$

$$\sigma_x = 0, \quad \tau_{xy} = 0, \quad w = 0 \tag{11.5}$$

It is assumed throughout that the edge lies along the line $x = 0$ and that the plate is situated on the side of negative values of $x$.

In the classical theory of plates the edge on which such boundary conditions apply is considered as a free edge in the case of (11.1), as fully fixed in cases (11.2) and (11.3) and as hinged in cases (11.4) and (11.5).

It is perhaps more natural to represent total fixity by the three-dimensional boundary conditions (11.2), and a hinged support by the three-dimensional boundary conditions (11.5), but for purposes of comparison, boundary conditions (11.3) and (11.4) will be used.

It can easily be shown that in the present problem the boundary conditions

$$\tau_{xz}|_{\xi=0} = 0, \qquad \partial\tau_{xz} / \partial\zeta|_{\xi=0} = 0$$

are equivalent to each other. In fact, in a solution which satisfies the second condition, $\tau_{xz}$ at $\xi = 0$ must be constant with respect to $\zeta$, and since this quantity is zero at $\zeta = \pm 1$, the first equality must be satisfied.

The equivalence of the boundary conditions

$$v|_{\xi=0} = 0, \qquad \partial v / \partial\zeta|_{\xi=0} = 0$$

can be proved in the same way, since $v$ is assumed to be an odd function of $\zeta$.

An obvious corollary of these conclusions is the equivalence of both of the two pairs of boundary conditions

$$(\tau_{xy} = 0,\ \tau_{xz} = 0), \qquad (v = 0,\ w = 0) \tag{11.6}$$

respectively to the following two pairs of boundary conditions

$$\left(\tau_{xy} = 0,\ \ \tau \equiv \frac{\partial\tau_{xy}}{\partial y} - h^{-1}\,\frac{\partial\tau_{xz}}{\partial\zeta} = 0\right) \quad \left(\gamma_{yz} \equiv h^{-1}\frac{\partial v}{\partial\zeta} + \frac{\partial w}{\partial y} = 0,\ w = 0\right) \tag{11.7}$$

**12.** It will be assumed that the stresses and displacements are composed of the sums of three terms corresponding to the basic iteration process and two variants of the auxiliary processes. In other words, we assume that the stresses and displacements can be expressed as follows:

$$\sigma_x = h^{-2}\sum h^{s-1}(\zeta\sigma_{x1}^{(s)} + \sigma_x^{*(s)}) + h^{-\lambda+1}\sum h^{s-1}(\sigma_{x\mathrm{I}}^{(s)} + \sigma_{x\mathrm{I}}^{*(s)}) +$$
$$+ h^{-\mu+1}\sum h^{s-1}(\sigma_{x\mathrm{II}}^{(s)} + \sigma_{x\mathrm{II}}^{*(s)})$$

$$\tau_{xy} = h^{-2}\sum h^{s-1}(\zeta\tau_{xy\mathrm{I}}^{(s)} + \tau_{xy}^{*(s)}) + h^{-\lambda}\sum h^{s-1}(\tau_{xy\mathrm{I}}^{(s)} + \tau_{xy\mathrm{I}}^{*(s)}) +$$
$$+ h^{-\mu+2}\sum h^{s-1}(\tau_{xy\mathrm{II}}^{(s)} + \tau_{xy\mathrm{II}}^{*(s)})$$

$$\tau_{xz} = h^{-1}\sum h^{s-1}(\zeta^2\tau_{xz2}^{(s)} + \tau_{xz0}^{(s)} + \tau_{xz}^{*(s)}) +$$
$$+ h^{-\lambda+1}\sum h^{s-1}(\tau_{xz\mathrm{I}}^{(s)} + \tau_{xz\mathrm{I}}^{*(s)}) + h^{-\mu+1}\sum h^{s-1}(\tau_{xz\mathrm{II}}^{(s)} + \tau_{xz\mathrm{II}}^{*(s)})$$

$$u = h^{-2}\sum h^{s-1}(\zeta u_1{}^{(s)} + u^{*(s)}) + h^{-\lambda+2}\sum h^{s-1}(u_{\rm I}{}^{(s)} + u_{\rm I}{}^{*\,(s)}) + $$
$$+ h^{-\mu+2}\sum h^{s-1}(u_{\rm II}{}^{(s)} + u_{\rm II}{}^{*(s)})$$
$$v = h^{-2}\sum h^{s-1}(\zeta v_1{}^{(s)} + v^{*(s)}) + h^{-\lambda+1}\sum h^{s-1}(v_{\rm I}{}^{(s)} + v_{\rm I}{}^{*(s)}) +$$
$$+ h^{-\mu+3}\sum h^{s-1}(v_{\rm II}^{(s)} + v_{\rm II}{}^{*(s)})$$
$$w = h^{-3}\sum h^{s-1}(w_0{}^{(s)} + w^{*(s)}) + h^{-\lambda+2}\sum h^{s-1}(w_{\rm I}{}^{(s)} + w_{\rm I}{}^{*(s)}) +$$
$$+ h^{-\mu+2}\sum h^{s-1}(w_{\rm II}{}^{(s)} + w_{\rm II}{}^{*(s)}) \qquad (12.1)$$

The summation here is everywhere carried out with respect to integer values of $s$, starting with $s - 1$.

Similarly, we can write down expressions for $\tau$ and $\gamma_{yz}$ appearing in (11.7)

$$\tau = h^{-2}\sum h^{s-1}(\zeta\tau_1{}^{(s)} + \tau^{*(s)}) + h^{-\lambda+2}\sum h^{s-3}\tau_1{}^{*(s)} -$$
$$- h^{-\mu}\sum h^{s-1}\frac{\partial}{\partial\zeta}(\tau_{xz{\rm II}}^{(s)} + \tau_{xz\,{\rm II}}^{*(s)}) + h^{-\mu+2}\sum h^{s-1}\frac{\partial}{\partial y}(\tau_{xy{\rm II}}^{(s)} + \tau_{xy\,{\rm II}}^{*(s)}) \qquad (12.2)$$
$$\gamma_{yz} = h^{-1}\sum h^{s-3}\gamma_{yz}^{*(s)} + h^{-\lambda}\sum h^{s-1}\frac{\partial}{\partial\zeta}(v_{\rm I}{}^{(s)} + v_{\rm I}{}^{*(s)}) +$$
$$+ h^{-\lambda+2}\sum h^{s-1}\frac{\partial}{\partial y}(w_{\rm I}{}^{(s)} + w_{\rm I}{}^{*(s)}) + h^{-\mu+2}\sum h^{s-1}(\gamma_{yz{\rm II}}^{(s)} + \gamma_{yz{\rm II}}^{*(s)})$$

Here we have taken into account that by virtue of (3.3), (6.1) to (6.3)

$$\tau_{\rm I}{}^{(s)} \equiv \frac{\partial\tau_{xy{\rm I}}^{(s)}}{\partial y} - \frac{\partial\tau_{xz{\rm I}}^{(s)}}{\partial\zeta} = 0, \qquad \gamma_{yz_1}^{(s)} \equiv v_1{}^{(s)} + \frac{\partial w_0{}^{(s)}}{\partial y} = 0$$

We have also made use of the notations

$$\frac{\partial v^{*(s)}}{\partial\zeta} + \frac{\partial w^{*(s)}}{\partial y} = \gamma_{yz}^{*(s)}, \qquad \frac{\partial v_{\rm II}{}^{(s)}}{\partial\zeta} + \frac{\partial w_{\rm II}{}^{(s)}}{\partial y} = \gamma_{yz{\rm II}}^{(s)}$$
$$\frac{\partial v_{\rm II}{}^{*(s)}}{\partial\zeta} + \frac{\partial w_{\rm II}{}^{*(s)}}{\partial y} = \gamma_{yz{\rm II}}^{*(s)}, \qquad \tau_1{}^{(s)} = \frac{\partial\tau_{xy1}^{(s)}}{\partial y} - 2\tau_{xz2}^{(s)} \qquad (12.3)$$
$$\tau^{*(s)} = \frac{\partial\tau_{xy}^{*(s)}}{\partial y} - \frac{\partial\tau_{xz}^{*(s)}}{\partial\zeta}$$

It should be remembered that quantities distinguished by an asterisk vanish for $s = 1, 2$, and therefore in (12.2), in the expressions

$$\sum h^{s-3}\gamma_{yz}^{*(s)}, \qquad \sum h^{s-3}\tau_{\mathrm{I}}^{*(s)}$$

the first two terms disappear, and these summations, as well as all the others, start from terms containing $h$ to the power zero.

On the right-hand sides of Formulas (12.1) and (12.2) the numbers $\lambda$ and $\mu$ are as yet unknown, and the method of imposing the boundary conditions will be as follows. Expressions (12.1) and (12.2) are substituted into the boundary conditions, the numbers $\lambda$ and $\mu$ are chosen in some way and the coefficients of equal powers of $h$ on the right- and left-hand sides of each boundary condition taken separately are equated.

As a result a succession of boundary relations is obtained, the form of which depends on the choice of the numbers $\lambda$ and $\mu$. The values of $\lambda$ and $\mu$ must be chosen in such a way that the succession of boundary relations is consistent with the differential equations which are satisfied in each approximation by quantities associated with some particular iteration process.

**13.** Let us apply this method to boundary conditions (11.1) to (11.5). From (11.1), putting $\lambda = 2$, $\mu = 2$, we obtain a succession of boundary relations

$$\zeta\sigma_{x1}^{(1)} = 0, \quad \zeta\sigma_{x1}^{(2)} + \sigma_{x\mathrm{I}}^{(1)} + \sigma_{x\mathrm{II}}^{(1)} = 0, \quad \zeta\sigma_{x1}^{(3)} + \sigma_{x1}^{*(3)} + \sigma_{x\mathrm{I}}^{(2)} + \sigma_{x\mathrm{II}}^{(2)} = 0, \ldots$$

$$\zeta\tau_{xy1}^{(1)} + \tau_{xy\mathrm{I}}^{(1)} = 0, \qquad \zeta\tau_{xy1}^{(2)} + \tau_{xy\mathrm{I}}^{(2)} = 0$$

$$\zeta\tau_{xy}^{(3)} + \tau_{xy}^{*(3)} + \tau_{xy\mathrm{I}}^{(3)} + \tau_{xy\mathrm{I}}^{*(3)} + \tau_{xy\mathrm{II}}^{(1)} = 0, \ldots \tag{13.1}$$

$$\zeta\tau_1^{(1)} - \frac{\partial\tau_{xz\mathrm{II}}^{(1)}}{\partial\zeta} = 0, \qquad \zeta\tau_1^{(2)} - \frac{\partial\tau_{xz\mathrm{II}}^{(2)}}{\partial\zeta} = 0$$

$$\zeta\tau_1^{(3)} + \tau^{*(3)} + \tau_{\mathrm{I}}^{*(3)} - \frac{\partial}{\partial\zeta}\left(\tau_{xz\mathrm{II}}^{(3)} + \tau_{xz\mathrm{II}}^{*(3)}\right) + \frac{\partial\tau_{xy\mathrm{II}}^{(1)}}{\partial y} = 0$$

Here the first pair of boundary conditions (11.6) have been replaced by the first pair of (11.7).

From (11.2), putting $\lambda = 1$, $\mu = 3$, we obtain

$$\zeta u_1^{(1)} = 0, \qquad \zeta u_1^{(2)} + u_{\mathrm{II}}^{(1)} = 0, \qquad \zeta u_1^{(3)} + u^{*(3)} + u_{\mathrm{II}}^{(2)} = 0, \ldots$$

$$\gamma_{yz}^{*(3)} + \frac{\partial v_{\mathrm{I}}^{(1)}}{\partial\zeta} + \gamma_{yz\mathrm{II}}^{(1)} = 0, \qquad \gamma_{yz}^{*(4)} + \frac{\partial v_{\mathrm{I}}^{(2)}}{\partial\zeta} + \gamma_{yz\mathrm{II}}^{(2)} = 0, \ldots \tag{13.2}$$

$$w_0^{(1)} = 0, \quad w_0^{(2)} = 0, \qquad w_0^{(3)} + w^{*(3)} + w_{\mathrm{II}}^{(1)} = 0, \ldots$$

Here the second pair of boundary conditions (11.6) has been replaced by the second pair of (11.7).

From (11.3), putting $\lambda = 2$, $\mu = 3$, we find that

$$\begin{gathered}\zeta u_{\mathrm{I}}^{(1)} = 0, \qquad \zeta u_{\mathrm{I}}^{(2)} + u_{\mathrm{II}}^{(1)} = 0, \quad \zeta u_{\mathrm{I}}^{(3)} + u^{*(3)} + u_{\mathrm{II}}^{(2)} + u_{\mathrm{I}}^{(1)} = 0, \ldots \\ \zeta \tau_{xy\mathrm{I}}^{(1)} + \tau_{xy\mathrm{I}}^{(1)} = 0, \qquad \zeta \tau_{xy\mathrm{I}}^{(2)} + \tau_{xy\mathrm{I}}^{(2)} + \tau_{xy\mathrm{II}}^{(1)} = 0, \ldots \qquad (13.3) \\ w_0^{(1)} = 0, \qquad w_0^{(2)} = 0, \qquad w_0^{(3)} + w^{*(3)} + w_{\mathrm{II}}^{(1)} = 0, \ldots\end{gathered}$$

From (11.4), putting $\lambda = 1$, $\mu = 3$, we obtain

$$\begin{gathered}\zeta \sigma_{x\mathrm{I}}^{(1)} + \sigma_{x\mathrm{II}}^{(1)} = 0, \ \zeta \sigma_{x\mathrm{I}}^{(2)} + \sigma_{x\mathrm{II}}^{(2)} = 0, \quad \zeta \sigma_{x\mathrm{I}}^{(3)} + \sigma_x^{*(3)} + \sigma_{x\mathrm{II}}^{(3)} + \sigma_{x\mathrm{II}}^{*(3)} + \sigma_{x\mathrm{I}}^{(1)} = 0, \ldots \\ \gamma_{yz}^{*(3)} + \frac{\partial v_{\mathrm{I}}^{(1)}}{\partial \zeta} + \gamma_{yz\mathrm{II}}^{(1)} = 0, \qquad \gamma_{yz}^{*(4)} + \frac{\partial v_{\mathrm{I}}^{(2)}}{\partial \zeta} + \gamma_{yz\mathrm{II}}^{(2)} = 0, \ldots \qquad (13.4) \\ w_0^{(1)} = 0, \qquad w_0^{(2)} = 0, \qquad w_0^{(3)} + w^{*(3)} + w_{\mathrm{II}}^{(1)} = 0, \ldots\end{gathered}$$

The boundary conditions here have been rearranged as in (11.2).

From (11.5), putting $\lambda = 2$, $\mu = 3$, we obtain

$$\begin{gathered}\zeta \sigma_{x\mathrm{I}}^{(1)} + \sigma_{x\mathrm{II}}^{(1)} = 0, \quad \zeta \sigma_{x\mathrm{I}}^{(2)} + \sigma_{x\mathrm{II}}^{(2)} + \sigma_{x\mathrm{I}}^{(1)} = 0, \\ \zeta \sigma_{x\mathrm{I}}^{(3)} + \sigma_x^{*(3)} + \sigma_{x\mathrm{II}}^{(3)} + \sigma_{x\mathrm{II}}^{*(3)} + \sigma_{x\mathrm{I}}^{(2)} = 0, \ldots \\ \zeta \tau_{xy\mathrm{I}}^{(1)} + \tau_{xy\mathrm{I}}^{(1)} = 0, \qquad \zeta \tau_{xy\mathrm{I}}^{(2)} + \tau_{xy\mathrm{I}}^{(2)} + \tau_{xy\mathrm{II}}^{(1)} = 0, \ldots \qquad (13.5) \\ w_0^{(1)} = 0, \qquad w_0^{(2)} = 0, \qquad w_0^{(3)} + w^{*(3)} + w_{\mathrm{II}}^{(1)} = 0, \ldots\end{gathered}$$

The compatibility of the boundary relations (13.1) to (13.5) is proved below.

**14.** Each succession of boundary relations (13.1 to (13.5) consists of equalities which form three groups corresponding to the number of boundary conditions. The first and third groups can be looked upon as equations defining the boundary values of quantities distinguished by the additional subscript II. An exception in certain cases occurs in one or two of the first equalities in each group where these quantities do not appear. For quantities distinguished by the subscript II the compatibility conditions (10.5) must be satisfied. This enables us to write two groups of supplementary relations for each of the boundary conditions (11.1) to (11.5).

For the boundary conditions (13.1) the supplementary relations are

$$\sigma_{x1}{}^{(1)} = 0, \quad A\left(-\zeta\sigma_{x1}^{(2)} - \sigma_{xI}^{(1)},\ \sigma_{xII}^{(1)}\right) \equiv -\frac{2}{3}\sigma_{x1}^{(2)} - \int_{-1}^{+1}\zeta\sigma_{xI}^{(1)}d\zeta = 0$$

$$A\left(-\zeta\sigma_{x1}^{(3)} - \sigma_{x1}^{*(3)} - \sigma_{xI}^{(2)},\ \sigma_{xII}^{(2)}\right) \equiv -\frac{2}{3}\sigma_{x1}^{(3)} - \int_{-1}^{+1}\zeta\left(\sigma_{x1}^{*(3)} + \sigma_{xI}^{(2)}\right)d\zeta, \ldots$$

$$A\left(\zeta\tau_1{}^{(1)},\ \frac{\partial\tau_{xzII}^{(1)}}{\partial\zeta}\right) \equiv \frac{2}{3}\tau_1{}^{(1)} = 0$$

$$A\left(\zeta\tau_1{}^{(2)},\ \frac{\partial\tau_{xzII}^{(2)}}{\partial\zeta}\right) \equiv -\frac{2}{3}\tau_1{}^{(2)} = 0, \ldots \tag{14.1}$$

$$A\left(\zeta\tau_1{}^{(3)} + \tau^{*(3)} + \tau_1{}^{*(3)} - \frac{\partial\tau_{xz\,II}^{*(3)}}{\partial\zeta},\ \frac{\partial\tau_{xzII}^{(3)}}{\partial\zeta}\right) \equiv$$

$$\equiv -\frac{2}{3}\tau_1^{(3)} + \int_{-1}^{+1}d\zeta\int_{-1}^{\zeta}\left(\tau^{*(3)} + \tau_1{}^{*(3)} - \frac{\partial\tau_{xzII}^{*(3)}}{\partial\zeta}\right)d\zeta = 0, \ldots$$

Here, in writing out the compatibility relations $A = 0$ we have made use of Formulas (10.2) and (10.4). Also, the first equality in each group of boundary relations, in which $\sigma_{xII}^{(s)}$ does not appear, has been re-written here after dividing by $\zeta$ (as is done in all subsequent supplementary relations).

For the boundary conditions (13.2) and (13.3) we obtain

$$u_1{}^{(1)} = 0, \qquad A\left(-\zeta u_1{}^{(2)},\ u_{II}{}^{(1)}\right) = 0$$

$$A\left(-\zeta u_1{}^{(3)} - u^{*(3)} - ju_I{}^{(1)},\quad u_{II}{}^{(2)}\right) = 0, \ldots \tag{14.2}$$

$$w_0{}^{(1)} = 0, \qquad w_0{}^{(2)} = 0, \qquad A\left(-w_0{}^{(3)} - w^{*(3)},\ w_{II}{}^{(1)}\right) = 0, \ldots$$

In the third of these equalities we must put $j = 0$ for boundary conditions (13.2) and $j = 1$ for boundary conditions (13.3).

For the boundary conditions (13.4) we obtain

$$A\left(-\zeta\sigma_{x1}^{(1)},\ \sigma_{xII}^{(1)}\right) \equiv -\frac{2}{3}\sigma_{x1}^{(1)} = 0, \qquad A\left(-\zeta\sigma_{x1}^{(2)},\ \sigma_{xII}^{(2)}\right) \equiv -\frac{2}{3}\sigma_{x1}^{(2)} = 0$$

$$A\left(-\zeta\sigma_{x1}^{(3)} - \sigma_x{}^{*(3)} - \sigma_{xII}^{*(3)} - \sigma_{xI}^{(1)},\ \sigma_{xII}^{(3)}\right) = -\frac{2}{3}\sigma_{x1}^{(3)} -$$

$$-\int_{-1}^{+1}\zeta\left(\sigma_x{}^{*(3)} + \sigma_{xII}^{*(3)} + \sigma_{xI}^{(1)}\right)d\zeta = 0\ldots \tag{14.3}$$

$$w_0{}^{(1)} = 0, \qquad w_0{}^{(2)} = 0, \qquad A\left(-w_0{}^{(3)} - w^{*(3)},\ w_{II}{}^{(1)}\right) = 0\ldots$$

and for boundary conditions (13.5) we find that

$$
\begin{aligned}
&A\left(-\zeta\sigma_{x\mathrm{I}}^{(1)},\ \sigma_{x\mathrm{II}}^{(1)}\right) \equiv -\frac{2}{3}\sigma_{x\mathrm{I}}^{(1)} = 0\\
&A\left(-\zeta\sigma_{x\mathrm{I}}^{(2)} - \sigma_{x\mathrm{I}}^{(1)},\ \sigma_{x\mathrm{II}}^{(2)}\right) \equiv -\frac{2}{3}\sigma_{x\mathrm{I}}^{(2)} - \int_{-1}^{+1}\zeta\sigma_{x\mathrm{I}}^{(1)}\,d\zeta = 0\\
&A\left(-\zeta\sigma_{x\mathrm{I}}^{(3)} - \sigma_x^{*(3)} - \sigma_{x\mathrm{II}}^{*(3)} - \sigma_{x\mathrm{I}}^{(2)},\ \sigma_{x\mathrm{II}}^{(2)}\right) \equiv\\
&\equiv -\frac{2}{3}\sigma_{x\mathrm{I}}^{(3)} - \int_{-1}^{+1}\zeta\left(\sigma_x^{*(3)} + \sigma_{x\mathrm{II}}^{*(3)} + \sigma_{x\mathrm{I}}^{(2)}\right)d\zeta = 0\\
&w_0^{(1)} = 0, \qquad w_0^{(2)} = 0, \quad A\left(-w_0^{(3)} - w^{*(3)},\ w_{\mathrm{II}}^{(1)}\right) = 0\ \ldots
\end{aligned}
\tag{14.4}
$$

**15.** In the series (12.1) and (12.2) quantities of the type $Q_i^{(s)}$ are functions of the two variables $(x, y)$, and for each particular value of $s$ satisfy a system of equations equivalent to one biharmonic equation. This means that $Q_i^{(s)}$ can be expressed for every value of $s$ in terms of a biharmonic function $B^{(s)}$ of the variables $(x, y)$.

Quantities of the type $Q^{*(s)}$ are polynomials in $\zeta$, the coefficients of which can be expressed by $Q^{(1)}, Q^{(2)}, \ldots, Q^{(s-2)}$. Since $Q^{(1)}, Q^{(2)}, Q^{(3)}, \ldots$ are determined successively (in numerical order of superscripts), it follows that in finding $Q^{(s)}$ or, what amounts to the same thing, in determining $B^{(s)}$, the quantities $Q^{*(s)}$ can be considered to be known.

Quantities of the type $R_{\mathrm{I}}^{(s)}$ can be expressed for every value of $s$ in terms of a biharmonic function $\Psi^{(s)}$ of the variables $(\xi, \zeta)$. Quantities of the type $R_{\mathrm{II}}^{(s)}$ can be expressed for every value of $s$ in terms of a biharmonic function $\Phi^{(s)}$ of the variables $(\xi, \zeta)$. Finally, quantities of the type $R_{\mathrm{I}}^{*(s)}$ and $R_{\mathrm{II}}^{*(s)}$ can be expressed, respectively, in terms of $R_{\mathrm{I}}^{(1)}, R_{\mathrm{I}}^{(2)}, \ldots, R_{\mathrm{I}}^{(s-2)}$ and $R_{\mathrm{II}}^{(1)}, R_{\mathrm{II}}^{(2)}, \ldots, R_{\mathrm{II}}^{(s-2)}$. This means that in finding $\Psi^{(s)}$ quantities of the type $R_{\mathrm{I}}^{*(s)}$ can be considered to be known, and likewise for quantities of the type $R_{\mathrm{II}}^{*(s)}$ in the determination of $\Phi^{(s)}$.

Thus we are able to find the functions $B^{(s)}$, $\Psi^{(s)}$, $\Phi^{(s)}$ the boundary conditions for which are formed by relations (13.1) to (13.5) and (14.1) to (14.4).

In determining $B^{(1)}, B^{(2)}, \ldots$ the boundary conditions can be obtained by using equalities of corresponding numbers from the first and second groups of supplementary relations (14.1) to (14.4). In finding $\Psi^{(1)}, \Psi^{(2)}, \ldots$ an equality of the corresponding number from the second group of boundary relations (13.1) to (13.5) is used for the boundary conditions. In determining $\Phi^{(1)}$, $\Phi^{(2)}$, ... the boundary conditions are obtained from equalities of corresponding numbers from the first and third groups of boundary relations (13.1) to (13.5), discarding those equalities that do not contain quantities of the type $Q_{II}^{(s)}$ (they are included in the supplementary relations).

For the boundary conditions (11.1), (11.3) and (11.5) the functions $B^{(s)}$, $\Psi^{(s)}$, $\Phi^{(s)}$ must be determined in the following order:

$$B^{(1)},\ \Psi^{(1)},\ B^{(2)},\ (\Psi^{(2)},\ \Phi^{(1)},\ B^{(3)}),\ \ldots \tag{15.1}$$

The fourth, fifth and sixth functions are bracketed, which means that in some cases the order in which these function are determined should be altered, and in other cases all three functions must be determined simultaneously.

In determining $B^{(1)}$, $\Psi^{(1)}$, $B^{(2)}$ the following equalities (written in square brackets for each of these three functions) form the boundary conditions:

for boundary conditions (11.1)

$$\begin{gathered} B^{(1)} \to [\sigma_{x1}^{(1)} = 0;\ \tau_1^{(1)} = 0], \qquad \Psi^{(1)} \to [\zeta\tau_{xyI}^{(1)} + \tau_{xyI}^{(1)} = 0] \\ B^{(2)} \to \left[\frac{2}{3}\sigma_{x1}^{(2)} + \int_{-1}^{+1} \zeta\sigma_{x1}^{(1)}\, d\zeta = 0;\ \tau_1^{(2)} = 0\right] \end{gathered} \tag{15.2}$$

for boundary conditions (11.3)

$$\begin{gathered} B^{(1)} \to [u^{(1)} = 0;\ w_0^{(1)} = 0], \qquad \Psi^{(1)} \to [\zeta\tau_{xyI}^{(1)} + \tau_{xyI}^{(1)} = 0], \\ B^{(2)} \to [A\ (-\ \zeta u_1^{(2)};\ u_{II}^{(1)}) = 0;\ w_0^{(2)} = 0] \end{gathered} \tag{15.3}$$

for boundary conditions (11.5)

$$\begin{gathered} B^{(1)} \to [\sigma_{x1}^{(1)} = 0;\ w_0^{(1)} = 0], \qquad \Psi^{(1)} \to [\zeta\tau_{xyI}^{(1)} + \tau_{xyI}^{(1)} = 0] \\ B^{(2)} \to \left[\frac{2}{3}\sigma_{x1}^{(2)} + \int_{-1}^{+1} \zeta\sigma_{x1}^{(1)}\, d\zeta = 0;\ w_0^{(2)} = 0\right] \end{gathered} \tag{15.4}$$

For boundary conditions (11.2) the functions $B^{(s)}$, $\Psi^{(s)}$, $\Phi^{(s)}$ must be determined in the following order:

$$B^{(1)},\ B^{(2)},\ B^{(3)},\ \Phi^{(1)},\ \Psi^{(1)},\ \ldots \tag{15.5}$$

and the boundary conditions are given by the equalities

$$B^{(1)} \to [u_1^{(1)} = 0;\ w_0^{(1)} = 0], \qquad B^{(2)} \to [A\,(-\zeta u_1^{(2)};\ u_{\mathrm{II}}^{(1)}) = 0;\ w_0^{(2)} = 0]$$

$$B^{(3)} \to [A\,(-\zeta u_1^{(3)} - u^{*(3)};\ u_{\mathrm{II}}^{(2)}) = 0,\ A\,(-w_0^{(3)} - w^{*(3)};\ w_{\mathrm{II}}^{(1)}) = 0]$$

$$\Phi^{(1)} \to [\zeta u_1^{(2)} + u_{\mathrm{II}}^{(1)} = 0;\ w_0^{(3)} + w^{*(3)} + w_{\mathrm{II}}^{(1)} = 0] \tag{15.6}$$

$$\Psi^{(1)} \to \left[\gamma_{yz}^{*(3)} + \frac{\partial v_{\mathrm{I}}^{(1)}}{\partial \zeta} + \gamma_{yz\mathrm{II}}^{(1)} = 0\right]$$

For the boundary conditions (11.4) all of the relations (15.6) are retained with the exception of the first boundary condition which applies to the function $B^{(3)}$. It has the form

$$A\,(-\zeta u_1^{(3)} - u^{*(3)} - u_{\mathrm{I}}^{(1)};\ u_{\mathrm{II}}^{(2)}) = 0$$

In this connection, only $B^{(1)}$ in (15.5) is determined independently; $B^{(2)}$, $B^{(3)}$, $\Phi^{(1)}$, $\Psi^{(1)}$ must be determined simultaneously.

**16.** We will now show that every approximate basic iteration process taken separately is equivalent to an analysis of the plate on the basis of the classical theory. We shall make use of the definitions

$$M_x^{(s)} = h^{s-3} \int_{+h}^{+h} z\sigma_{xi}^{(s)}\, dz \quad (xy), \qquad H^{(s)} \doteq h^{s-3} \int_{-h}^{+h} z\tau_{xyi}^{(s)}\, dz$$

$$V_x^{(s)} = h^{s-2} \int_{-h}^{+h} \tau_{xzi}^{(s)}\, dz \quad (xy), \qquad w^{(s)} = h^{s-4} w_0^{(s)} \tag{16.1}$$

The quantities $M_x^{(s)}$, $M_y^{(s)}$, $H^{(s)}$, $V_x^{(s)}$, $V_y^{(s)}$ have an obvious physical meaning: they are in fact the moments and shear forces produced by stresses corresponding in the $s$th approximation of the basic iteration process to the general solution of the homogeneous equations.

We replace the functions $\sigma_{xi}^{(s)}$, $\sigma_{yi}^{(s)}$, $\tau_{xyi}^{(s)}$, $\tau_{xzi}^{(s)}$, $\tau_{yzi}^{(s)}$ in (16.1) by their expressions given by (3.2); taking into account (2.1) and the last of formulas (3.6), we obtain

$$M_x^{(s)} = \frac{2}{3} h^{s-1} \sigma_{x1}^{(s)} \quad (xy), \qquad H^{(s)} = \frac{2}{3}\, h^{s-1} \tau_{xy1}^{(s)}$$

$$V_x = -\frac{4}{3}\, h^{s-1} \tau_{xz}^{(s)} - h^{s-1} \int_{-1}^{+1} \tau_{xz}^{*(s)}\, d\zeta \quad (xy) \tag{16.2}$$

Substituting this result into (3.3), we obtain:

$$M_x^{(s)} = -\frac{2Eh^3}{3(1-\nu^2)}\left(\frac{\partial^2 w^{(s)}}{\partial x^2} + \nu \frac{\partial^2 w^{(s)}}{\partial y^2}\right) \quad (xy), \qquad H^{(s)} = -\frac{2Eh^3}{3(1+\nu)}\frac{\partial^2 w^{(s)}}{\partial x \partial y}$$

$$V_x^{(s)} = \frac{\partial M_x^{(s)}}{\partial x} + \frac{\partial H^{(s)}}{\partial y} - h^{s-1}\int_{-1}^{+1} \tau_{xz}^{*(s)} d\zeta \quad (xy) \tag{16.3}$$

$$\frac{\partial^2 M_x^{(s)}}{\partial x^2} + \frac{\partial^2 H^{(s)}}{\partial x \partial y} + \frac{\partial^2 M_y^{(s)}}{\partial y^2} = 4h^{s-1}\sigma_{z3}^{(s)}$$

In making the $s$th approximation the quantities with an asterisk and a superscript $(s-1)$ can be taken as known. In addition, taking into account formulas (3.6), we can also consider $\tau_{xz}^{*(s)}$, $\tau_{yz}^{*(s)}$ and $\sigma_{z3}^{(s)}$ in (16.3) as known quantities. Thus Expressions (16.3) are the equations of the classical theory of plates. In these expressions the terms containing $\tau_{xz}^{*(s)}$, $\tau_{yz}^{*(s)}$ and $\sigma_{z3}^{(s)}$ represent the externally applied forces and moments respectively.

For $s = 1$ we have that $\tau_{xz}^{*(s)} = 0$, $\tau_{yz}^{*(s)} = 0$, $4\sigma_{z3}^{(s)} = p$ and the conditional load applied to the plate coincides with that considered in an analysis based on the classical theory.

For $s = 2$ we have that $\tau_{xz}^{*(s)} = 0$, $\tau_{yz}^{*(s)} = 0$, $\sigma_{z3}^{*(s)} = 0$ and the conditional load vanishes. For $s > 2$ the conditional forces and moments in the $s$th approximation depend on the stresses of the $(s-2)$th approximation. The conditional load intensity diminishes as $s$ increases according to the law $h^{s-1}$.

The boundary conditions required for finding the biharmonic function $B^{(1)}$ are given by (15.2) to (15.4) and (15.6). In these equalities $\tau_1^{(1)}$ can be expressed according to Formula (12.3) and $u_1^{(1)}$ can be written in terms of $w_0^{(1)}$ with the aid of (3.3). Taking this into account and making use of Formulas (16.2), we obtain:

the boundary conditions corresponding to a free edge

$$M_x^{(1)} = 0, \qquad V_x^{(1)} + \partial H^{(1)} / \partial y = 0$$

the boundary conditions corresponding to a fully fixed edge

$$\partial w^{(1)} / \partial x = 0, \qquad w^{(1)} = 0$$

and the boundary conditions corresponding to a simply-supported edge

$$M_x^{(1)} = 0, \qquad w^{(1)} = 0$$

It follows that the first approximation of the basic iteration process is equivalent to the classical theory of plates, in the identity not only of the differential equations, but also of the boundary conditions.

**17.** The most significant corollary of the preceding results is that the state of stress set up in a thin plate in bending is composed of a basic state of stress, a state of stress due to edge torsion and a state of stress due to plane deformation at the edges.

The basic state of stress corresponds to the basic iteration process. In general, it covers the whole plate and as a first approximation coincides with the state of stress corresponding to the hypotheses of the classical theory of plates. The states of stress due to edge torsion and plane deformation at the edges correspond to the first and second variants of the auxiliary iteration process. They have only a local effect near the edges of the plate or near other lines of distortion and as a first approximation coincide, respectively, with the states of stress set up by torsion or plane deformation of a narrow strip along the given line of distortion.

All of these states can be found as a first approximation by making use of the usual physical hypotheses, and with the aid of a basic iteration process and two variants of the auxiliary process they can be found for sufficiently small values of $h$ to any degree of accuracy (this statement is conditional, since the present paper does not cover the question of evaluating errors).

The problem of formulating various approximate methods for solving problems in the theory of plates can now be treated as one of finding a certain number of approximations in the iteration processes described above. In particular, the classical theory from this point of view can be looked upon as an approximate method based on the application of one basic iteration process for which only the first approximation is made. With this approach it is necessary to take into consideration the auxiliary iteration processes, i.e. iteration processes based on the integration of differential equations in which one of the independent variables is $\zeta$. This is the principal difference between the suggested approach and those which have been adopted so far. These have always led to the integration of equations with independent variables $(x, y)$ defining a point in the middle plane. In this connection, it would now be difficult to compare the proposed method with methods such as that suggested by Reissner [2]. However, a purely qualitative coincidence will be noted. The equations obtained in making the classical theory more exact contain additional integrals corresponding to rapidly damped states of stress.

**18.** Without going into the details of actually deriving the functions $B^{(s)}$, $\Psi^{(s)}$, $\Phi^{(s)}$, we shall conclude with a few remarks of a general nature.

We obtained the Formulas (2.3), (2.4) in which, as was shown in Section 3, we must put $\kappa = 0$. These relations define the asymptote of the basic state of stress, i.e. the rate of increase or decrease of its components of stress or displacement as $h$ tends to zero. The main stresses (those which increase more rapidly than the others) of the basic state of stress are $\sigma_x$, $\tau_{xy}$, $\sigma_y$. They increase as $h^{-2}$. We shall write this as follows:

$$\sigma_x, \quad \tau_{xy}, \quad \sigma_y \sim h^{-2} \tag{18.1}$$

The asymptote of the states of stress due to edge torsion and plane deformation at the edges is defined by Formulas (4.2, (4.3) and (4.2), (4.4), respectively. It is found from these relations that in the state of stress due to edge torsion the main stresses are $\tau_{xy}$, $\tau_{yz}$, where

$$\tau_{xy}, \quad \tau_{yz} \sim h^{-\lambda} \tag{18.2}$$

and in the state of stress due to plane deformation at the edges the main stresses are $\sigma_x$, $\sigma_y$, $\tau_{xz}$, $\sigma_z$, where

$$\sigma_x, \ \sigma_y, \ \tau_{xz}, \ \sigma_z \sim h^{-\mu+1} \tag{18.3}$$

Comparing (18.1), (18.2) and (18.3), we conclude that with $\lambda = 2$ and $\mu = 3$ the main stresses in the states of stress due to edge torsion and plane deformation at the edges are not of the same order as the main stresses of the basic state of stress. But it was shown in Section 13 that at least one of the quantities $\lambda$, $\mu$ assumes these values for any of the boundary conditions considered above. This means that the edge stresses (or, in general, the stresses near lines of disturbance) cannot be found even to a first approximation by means of one basic iteration process, or, consequently, by means of the classical theory.

In order to find more accurate values of stresses at points distant from the edges by the proposed method, it is sufficient to make the first two approximations in the basic iteration process. In most cases it is considerably more difficult to find the edge stresses even to a very crude approximation. To do so, it is necessary to find the first approximation of all three iteration processes, i.e. to determine $B^{(1)}$, $\Psi^{(1)}$, $\Phi^{(1)}$, but the functions $\Psi^{(1)}$, $\Phi^{(1)}$, as is shown in Section 15, are rather far apart in the successions which determine the order of finding $B^{(s)}$, $\Psi^{(s)}$, $\Phi^{(s)}$.

The boundary conditions (11.2) and (11.3), which are different from the point of view of the three-dimensional theory of elasticity, must in the classical theory of plates be treated as the same, as conditions of full fixity. Similarly, (11.4) and (11.5) are two distinct three-dimensional examples of conditions of hinged support. It can be shown that in the succession of functions $B^{(s)}$, $\Psi^{(s)}$, $\Phi^{(s)}$ only $B^{(1)}$ is independent of the choice of variant of the three-dimensional boundary conditions corresponding to a given condition of fixity. This means that only within the framework of the classical theory of plates can we use such general concepts as a fully fixed edge, a hinged support, etc. In deriving methods which aim at greater accuracy these concepts must be made far more definite.

In order to simplify the computations in the auxiliary iteration pfocesses it was assumed that an edge of the plate lies in the plane $x = 0$. It is not difficult, however, to generalize this to the case when the edge is in any position, since the main results can easily be formulated in terms which are independent of the chosen coordinate system.

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